Inner and Outer Approximations of Star-Convex Semialgebraic Sets

James Guthrie

***Abstract*— We consider the problem of approximating a semialgebraic set with a sublevel-set of a polynomial function. In this setting, it is standard to seek a minimum volume outer approximation and/or maximum volume inner approximation. As there is no known relationship between the coefficients of an arbitrary polynomial and the volume of its sublevel sets, previous works have proposed heuristics based on the determinant and trace objectives commonly used in ellipsoidal fitting. For the case of star-convex semialgebraic sets, we propose a novel objective which yields both an outer and an inner approximation while minimizing the ratio of their respective volumes. This objective is scale-invariant and easily interpreted. Numerical examples are given which show that the approximations obtained are often tighter than those returned by existing heuristics. We also provide methods for establishing the star-convexity of a semialgebraic set by finding inner and outer approximations of its kernel.**

I. INTRODUCTION

Consider a compact, semialgebraic set $\mathcal{X} \subset \mathbb{R}^n$ given by the intersection of the 1-sublevel sets of m polynomial functions $g_i(x) \in \mathbb{R}[x]$:

$$\mathcal{X} = \{x \,|\, g_i(x) \leq 1,\, i \in [m]\} \tag{1}$$

Semialgebraic sets arise naturally in many control applications. For example, the set of coefficients for which a polynomial is Schur or Hurwitz stable is given by a semialgebraic set. These sets are often complicated and cumbersome to analyze. As such, it is common to seek simpler representations which closely approximate the set but are more amenable to further analysis [1]. Examples of "simple" representations include hyperrectangles and ellipsoids.

A number of publications have explored the use of sum-of-squares (SOS) optimization for approximating a semialgebraic set with a simpler representation [2]–[8]. The most common parameterization is to seek a SOS polynomial whose 1-sublevel set $\mathcal{F} = \{x \,|\, f(x) \leq 1\}$ provides either an inner ($\mathcal{F} \subseteq \mathcal{X}$) or outer ($\mathcal{F} \supseteq \mathcal{X}$) approximation of the set $\mathcal{X}$. In this formulation, an open question is the choice of the objective function. For outer (resp. inner) approximations, a natural objective is to minimize (resp. maximize) the volume of the 1-sublevel set. For an ellipsoid $\mathcal{E} = \{x \,|\, x^T Ax + b^T x + c \leq 1\}$ where $A \succeq 0$, the volume is proportional to $\det A^{-1}$. Using the logarithmic transform, ellipsoidal volume minimization can be posed as the convex objective $-\log\det A$ [9]. More generally, in the case of homogeneous polynomials it is possible to find the minimum volume outer approximation by solving a hierarchy of semidefinite programs [10].

J. Guthrie is with the Department of Electrical and Computer Engineering, 3400 N. Charles Street, Johns Hopkins University, Baltimore, MD 21218, USA. Email: jguthri6@jhu.edu

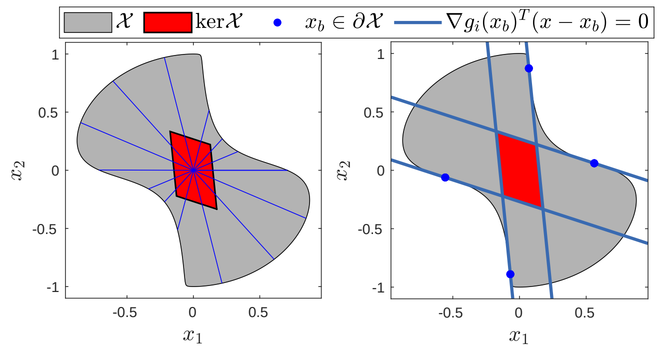

Fig. 1. Star-convex set $\mathcal{X}$ and its kernel (left). The kernel is a convex set given by the linearized active constraint $g_i(x_b) = 1$ defining $\partial\mathcal{X}$ (right).

While ellipsoids and homogeneous polynomials offer established techniques for approximating a set, they have inherent symmetry. Thus they are not ideal candidates for approximating general, asymmetric shapes. General polynomials offer a more flexible basis for approximating sets. The caveat is that we lack expressions for computing the volume of the 1-sublevel set as a function of the polynomial coefficients. The most common approach is to mimic the determinant ([2], [4]) or trace [1] objectives used in ellipsoidal fitting. While these methods often yield qualitatively good approximations, they suffer from a lack of interpretability as the objective generally has no explicit relationship to the volume of the 1-sublevel set beyond upper bounding it in some cases [1]. Thus assessing the quality of the solution requires post-processing by either 1) numerically computing the resulting volume or 2) plotting the resulting set for qualitative assessment.

A. Contributions

In this paper we propose a new approach for finding inner and outer approximations of semialgebraic sets using SOS optimization. Our method is tailored to cases in which the set is star-convex. To our knowledge, star-convexity has not been explored in the SOS literature with the exception of [11]. Our contributions are as follows:

- We propose and justify an algorithm based on SOS optimization for jointly finding an inner and outer approximation of a star-convex semialgebraic set by minimizing the volume of the outer approximation relative to the volume of the inner approximation. We provide numerical examples showing that this heuristic tends to yield better approximations than existing methods.
- We provide algorithms for establishing the star-convexity of a semialgebraic set by finding inner and outer approximations of its kernel.

The rest of the paper is organized as follows. Section II formally defines the problem we address and reviews the notion of star-convexity. Section III surveys the existing volume heuristics for SOS-based set approximation. Section IV proposes a new volume heuristic for finding outer and inner approximations. Section V provides methods for determining the star-convexity of a set by approximating its kernel. Section VI provides numerical examples. Section VII concludes the paper.

B. Notation

Let $\mathbb{Z}^+$ denote the set of positive integers. Let $S^{n-1} := \{x \in \mathbb{R}^n \,|\, \|x\| = 1\}$. Let $i \in [k] := \{1, \ldots, k\}$. The notation $P \succ 0$ ($P \succeq 0$) indicates that the symmetric matrix P is positive definite (resp. positive semidefinite). Given a compact set $\mathcal{X} \subset \mathbb{R}^n$, its volume (formally, Lebesgue measure) is denoted vol $\mathcal{X} := \int_{\mathcal{X}} dx$. Let $\sigma_{\mathcal{X}}(c) := \max_{x\in\mathcal{X}} c^T x$ denote the support function of $\mathcal{X}$ where $c \in S^{n-1}$. Given sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$ the (bi-directional) Hausdorff distance is $d_H(\mathcal{A}, \mathcal{B}) := \max(h(\mathcal{A}, \mathcal{B}), h(\mathcal{B}, \mathcal{A}))$ where $h(\mathcal{A}, \mathcal{B}) := \max_{a\in\mathcal{A}} \min_{b\in\mathcal{B}} \|a - b\|_2$.

The α-sublevel set of a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is $\{x \in \mathbb{R}^n \,|\, f(x) \leq \alpha\}$. For $x \in \mathbb{R}^n$, let $\mathbb{R}[x]$ denote the set of polynomials in x with real coefficients. Let $\mathbb{R}_d[x]$ denote the set of all polynomials in $\mathbb{R}[x]$ of degree less than or equal to d. A polynomial $p(x) \in \mathbb{R}[x]$ is a SOS polynomial if there exists polynomials $q_i(x) \in \mathbb{R}[x], i \in [j]$ such that $p(x) = q_1^2(x) + \ldots + q_j^2(x)$. We use $\sum[x]$ to denote the set of SOS polynomials in x. A polynomial of degree $2d$ is a SOS polynomial if and only if there exists $P \succeq 0$ (the Gram matrix) such that $p(x) = z(x)^T P z(x)$ where $z(x)$ is the vector of all monomials of x up to degree d [12]. To minimize notational clutter, we will sometimes list a polynomial $f(x)$ as a decision variable where it is implicitly understood that a monomial basis is specified by the user and a matrix P is introduced as a decision variable such that $f(x) = z(x)^T P z(x)$.

II. Problem Statement

Definition 1 (Star-Convex)**.** A set $\mathcal{S} \subseteq \mathbb{R}^n$ is star-convex if it has a non-empty kernel, where the kernel is defined as:

$$\ker \mathcal{S} = \{x \,|\, tx + (1-t)y \in \mathcal{S} \,\forall\, t \in [0,1], y \in \mathcal{S}\} \tag{2}$$

Intuitively, the kernel is the set of points in $\mathcal{S}$ from which one can "see" all of $\mathcal{S}$. It is straight-forward to show that the kernel is convex. Further, if $\mathcal{S}$ is convex then $\ker \mathcal{S} = \mathcal{S}$.

We will be interested in approximating the set (1) for the case in which it is star-convex with respect to the origin.

Problem 1 (Star-Convex Set Approximation)**.** Given a compact, semialgebraic set $\mathcal{X}$ with $0 \in \text{int}\mathcal{X} \cap \ker\mathcal{X}$ and $d \in \mathbb{Z}^+$ find a polynomial $f_o(x) \in \mathbb{R}_d[x]$ ($f_i(x) \in \mathbb{R}_d[x]$) whose 1-sublevel set $\mathcal{F}_o$ ($\mathcal{F}_i$) is of minimum (maximum) volume and is an outer (inner) approximation of $\mathcal{X}$.

$$\min_{f_o(x)\in\mathbb{R}_d[x]} \text{vol } \mathcal{F}_o \text{ s.t. } \mathcal{X} \subseteq \mathcal{F}_o \tag{3}$$

$$\max_{f_i(x)\in\mathbb{R}_d[x]} \text{vol } \mathcal{F}_i \text{ s.t. } \mathcal{F}_i \subseteq \mathcal{X} \tag{4}$$

In the sequel, we would like to approximate the kernel of the set $\mathcal{X}$ with a polytope.

Problem 2 (Kernel Approximation)**.** Given a semialgebraic set $\mathcal{X} \subset \mathbb{R}^n$ find a polytope $\mathcal{K}_o$ ($\mathcal{K}_i$) of minimum (maximum) volume that is an outer (inner) approximation of $\ker\mathcal{X}$.

$$\min \text{ vol } \mathcal{K}_o \text{ s.t. } \ker\mathcal{X} \subseteq \mathcal{K}_o \tag{5}$$

$$\max \text{ vol } \mathcal{K}_i \text{ s.t. } \mathcal{K}_i \subseteq \ker\mathcal{X} \tag{6}$$

III. Existing Volume Heuristics for Set Approximation

We first review existing heuristics for finding inner and outer approximations of a semialgebraic set $\mathcal{X}$ defined as in (1) using SOS optimization. Each of these methods finds an even-degree polynomial $f(x) \in \mathbb{R}_d[x]$ where $d \in \mathbb{Z}^+$ is specified by the user. The polynomial is parameterized as $f(x) = z(x)^T P z(x)$, where $z(x) \in \mathbb{R}_{d/2}[x]$ is a monomial basis with m terms and $P \in \mathbb{R}^{m\times m}$ is a symmetric matrix decision variable. The variations between the methods largely relate to the objective applied to the matrix P and whether it must be positive semidefinite (PSD). For general polynomials, there is no known relationship between P and the volume of the sublevel sets thus the following objectives are all heuristics in some sense. The reader is referred to the given references for further detail.

A. Determinant Maximization ($-\det P$)

In [2], the authors propose maximizing the determinant of the Hessian $\nabla^2 f(x)$ of SOS polynomials. If f is a polynomial of degree 2, this reduces to the ellipsoidal objective $-\det A$ for $\mathcal{E} = \{x \,|\, x^T A x + b^T x + c \leq 1\}, A \succeq 0$. A limitation of this approach is that the Hessian must be PSD and therefore the outer approximation is inherently convex, making it ill-suited to approximating non-convex shapes.

In [4], the authors propose performing determinant maximization directly on the Gram matrix P of the SOS polynomial. As the Hessian is no longer required to be PSD, this allows non-convex outer approximations to be found.

B. Inverse Trace Minimization ($\text{tr}P^{-1}$)

The determinant maximization objective minimizes the product of the eigenvalues of P^{-1}. In [4], the authors propose an alternative heuristic of minimizing the sum of the eigenvalues of P^{-1}. This requires introducing an additional PSD decision variable V and imposing the constraint $V \succeq P^{-1}$. Using the Schur complement this can be written as a block matrix constraint involving V and P (vice P^{-1}). The objective min tr$V$ then indirectly minimizes the sum of the eigenvalues of P^{-1}.

C. l_1 Minimization

In [1] the authors propose a volume heuristic based on minimizing the l_1 norm of a polynomial evaluated over a bounding box $\mathcal{B} \supseteq \mathcal{X}$. Using hyperrectangles as bounding boxes, one can integrate the polynomial and the resulting objective $l_1(f(x)) := \int_{\mathcal{B}} f(x)\,dx$ is linear in terms of P. The outer approximation then consists of the intersection of the 1-superlevel set of $f(x)$ and the bounding box $\mathcal{B}$.

$$\mathcal{X} \subseteq (\mathcal{B} \cap \{x \,|\, f(x) \geq 1\}) \tag{7}$$

This is in contrast to the other objectives which do not rely on bounding boxes as part of the set approximation.[1] In this setting, $f(x)$ is approximating the indicator function of $\mathcal{X}$ over a compact set $\mathcal{B}$. Given $\mathcal{B}$ is compact, it is possible to prove convergence of $f(x)$ to the true indicator function in the limit (as $d \to \infty$) by leveraging the Stone-Weierstrass theorem. However, the rate of convergence remains unknown and for a given degree d, other objectives may return a better outer approximation. One distinct advantage of this method is that inner approximations can also be found by outer approximating the complement of $\mathcal{X}$.

IV. Inner and Outer Approximations of Star-Convex Sets

We now propose a new volume heuristic for solving Problem 1. Recall the following property for the volume of a scaled set which can be shown via a change of variables.

Lemma 1. *Let $\mathcal{X} \subset \mathbb{R}^n$. Let $s\mathcal{X} = \{sx \,|\, x \in \mathcal{X}\}$ denote the scaled set where $s \geq 0$. Then* vol $s\mathcal{X} = s^n \cdot \text{vol}\mathcal{X}$.

The following Lemma which is easily shown provides conditions under which we can scale an inner approximation to become an outer approximation of a given compact set.

Lemma 2. *Let $\mathcal{X}, \mathcal{F}$ be compact sets in $\mathbb{R}^n$ such that $\mathcal{F} \subseteq \mathcal{X}$. Let $0 \in \text{int}\,\mathcal{F}$. Then there exists a scaling $s \geq 1$ such that $\mathcal{X} \subseteq s\mathcal{F}$.*

Taken together, these lemmas suggest an intuitive heuristic for jointly finding an inner and outer approximation of a compact set $\mathcal{X}$ by minimizing the scaling required to turn an inner approximation into an outer approximation. Although applicable to approximating any compact set containing the origin in its interior, this heuristic is best suited to approximating star-convex sets in which $0 \in \text{int}\mathcal{X} \cap \ker \mathcal{X}$ as visualized in Figure 2.

In our formulation, we seek a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ whose 1-sublevel set $\mathcal{F} = \{x \,|\, f(x) \leq 1\}$ provides an inner approximation of $\mathcal{X}$. We turn this into a condition involving the complement of $\mathcal{X}$:

$$\mathcal{F} \subseteq \mathcal{X} \iff f(x) > 1 \,\forall\, x \in \mathcal{X}^{\mathsf{c}} \tag{8}$$

As optimization methods require non-strict inequalities, we will approximate this condition by introducing a small constant $\epsilon > 0$ and working with the closure of the complement of $\mathcal{X}$. Define the following:

$$\bar{\mathcal{X}} = \bigcup_{i \in [m]} \{x \,|\, g_i(x) \geq 1\} \tag{9}$$

[1] One application of approximating semialgebraic sets is to yield a single sufficient condition for ensuring $x \notin \mathcal{X}$, which can be incorporated into a nonlinear optimization problem (e.g. obstacle avoidance in motion planning [7]). The presence of the bounding box in the resulting set description would require logical constraints to represent $(f(x) < 1 \vee x \notin \mathcal{B}) \implies x \notin \mathcal{X}$ which are generally unsupported in nonlinear optimization solvers.

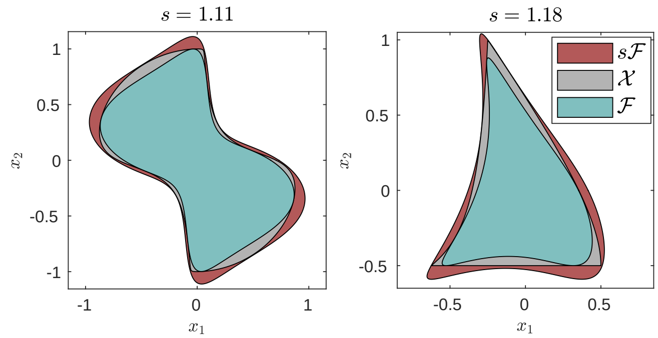

Fig. 2. 4th-order approximations of examples A (left) and B (right)

Then we can use the following approximation of (8)

$$\mathcal{F} \subseteq \text{int}\mathcal{X} \Leftarrow f(x) \geq 1 + \epsilon \,\forall\, x \in \bar{\mathcal{X}} \tag{10}$$

Next, we scale the set $\mathcal{F}$ by a scaling variable $s > 1$ to obtain an outer approximation $s\mathcal{F}$.

$$s\mathcal{F} \supseteq \mathcal{X} \iff f(\tfrac{x}{s}) \leq 1 \,\forall\, x \in \mathcal{X} \tag{11}$$

Combining the above we arrive at the following:

$$\begin{aligned} \min_{f(x),\, s} \quad & s \\ \text{s.t.} \quad & \\ & f(x) \geq 1 + \epsilon \,\forall\, x \in \bar{\mathcal{X}}, \\ & f(\tfrac{x}{s}) \leq 1 \qquad \forall\, x \in \mathcal{X}, \end{aligned} \tag{12}$$

Remark 1. From Lemma 1, we have the following relation:

$$\frac{\text{vol } s\mathcal{F}}{\text{vol } \mathcal{F}} = s^n \tag{13}$$

Thus by minimizing s we minimize the ratio of the outer approximation volume to the inner approximation volume.

We parameterize $f(x)$ as $f(x) = z(x)^T P z(x)$ where $z(x)$ is a monomial basis chosen by the user and P is a symmetric matrix of appropriate dimension. We introduce SOS polynomials $\lambda_i(x), \mu_i(x), i \in [m]$ and apply the $\mathcal{S}$-procedure to replace the set-containment conditions with sufficient SOS conditions [12]. If s is left as a decision variable, we would have bilinear terms involving the coefficients of $f(x)$ and s. As is common in the SOS literature, we overcome this by performing a bisection over s in which we solve a feasibility problem at each iteration as given by (14). Algorithm 1 details the bisection method.

Optimization Problem: FindApprox$(s, \mathcal{X}, z(x))$

$$\begin{aligned} \min_{f(x),\, \lambda_i(x),\, \mu_i(x)} \quad & 0 \\ \text{s.t.} \quad & \\ & f(x) - (1+\epsilon) - \lambda_i(x)(g_i(x) - 1) \in \textstyle\sum[x], i \in [m], \\ & 1 - f(\tfrac{x}{s}) - \sum_{i=1}^{m} \mu_i(x)(1 - g_i(x)) \in \textstyle\sum[x], \\ & \lambda_i(x), \mu_i(x) \in \textstyle\sum[x], \quad i \in [m], \end{aligned} \tag{14}$$

Algorithm 1 Inner and Outer Approximation of $\mathcal{X}$

Input: $\mathcal{X} \subset \mathbb{R}^n$, $z(x) \in \mathbb{R}[x]$, $s_{tol} > 0$
Output: $\mathcal{F}, s\mathcal{F}$ s.t. $\mathcal{F} \subseteq \mathcal{X} \subseteq s\mathcal{F}$

```
s_ub ← 1 + s_tol, s_lb ← 1
while FindApprox(s_ub, X, z(x)) = Infeasible do
    s_lb ← s_ub
    s_ub ← 2 s_ub
while s_ub − s_lb > s_tol do
    s_try ← 0.5(s_ub + s_lb)
    if FindApprox(s_try, X, z(x)) = Infeasible then
        s_lb ← s_try
    else
        s_ub ← s_try
return FindApprox(s_ub, X, z(x))
```

Remark 2. We note that the objective is scale-invariant. Assume we are given a solution pair $(f^*(x), s^*)$ that defines an outer and inner approximation of a set $\mathcal{X}$. If we scale $\mathcal{X}$ by $\alpha > 0$, the solution pair $(f^*(\frac{x}{\alpha}), s^*)$ defines the new approximation, where the objective value remains unchanged.

Remark 3. If $\mathcal{F}$ is convex we can relate the scaling s to the Hausdorff distance between the approximations.

Lemma 3. *Let $\mathcal{F} \subset \mathbb{R}^n$ be a convex, compact set and $s \geq 1$. Then the following holds:*

$$d_H(s\mathcal{F}, \mathcal{F}) = (s-1) \cdot \max_{x \in \mathcal{F}} \|x\|_2 \tag{15}$$

Proof. See appendix.

Remark 4. Existing approaches for finding inner or outer approximations with SOS optimization generally require $f(x)$ be SOS. Our formulation is slightly less restrictive in that $f(x)$ can be negative for $x \in \mathcal{X}$. Therefore we search over a larger candidate set of polynomials. In our initial experiments we have found adding the constraint $f(x) \in \sum[x]$ does not noticeably degrade the quality of our approximations. However, without this constraint, the resulting polynomial is often significantly sparser which is beneficial for compactness of representation and speed of evaluation.

V. Sampling-Based Approximations of the Kernel

Algorithm 1 assumed the set $\mathcal{X}$ contained the origin in its kernel. In this section we provide algorithms for finding polytopic approximations of $\ker \mathcal{X}$. These methods can be used to find a point $x^* \in \text{int}\mathcal{X} \cap \ker\mathcal{X}$ if one exists. We can then apply Algorithm 1 to the translated set $\{x - x^* \mid x \in \mathcal{X}\}$.

It will be convenient to represent the boundary of $\mathcal{X}$ in terms of the inequality that is active. Define the following:

$$\partial\mathcal{X}_i = \{x \mid g_i(x) = 1, g_j(x) \leq 1, j \in [m] \setminus i\} \tag{16}$$

The boundary of $\mathcal{X}$ is given by the union.

$$\partial\mathcal{X} = \bigcup_{i \in [m]} \partial\mathcal{X}_i \tag{17}$$

Lemma 4. *Let $\mathcal{X}$ be a semialgebraic set as defined in* (1)*. The kernel of $\mathcal{X}$ is given by the following semialgebraic set:*

$$\ker\mathcal{X} = \{x_k \mid \nabla g_i(x_b)^T (x_k - x_b) \leq 0 \, \forall \, x_b \in \partial\mathcal{X}_i, i \in [m]\}$$

Proof. $\Rightarrow$: Assume $x_k \in \ker\mathcal{X}$ but there exists a point $x_b \in \partial\mathcal{X}_i$ for some $i \in [m]$ such that $\nabla g_i(x_b)^T(x_k - x_b) > 0$. Recall the definition of the directional derivative:

$$\lim_{t \to 0} \frac{g_i(tx_k + (1-t)x_b) - g_i(x_b)}{t} = \nabla g_i(x_b)^T (x_k - x_b)$$

Noting that $g_i(x_b) = 1$ and $\nabla g_i(x_b)^T(x_k - x_b) > 0$ yields

$$\lim_{t \to 0} \frac{g_i(tx_k + (1-t)x_b) - 1}{t} > 0$$

This implies there exists an open interval $t \in (0, \alpha), \alpha > 0$ in which $g_i(tx_k + (1-t)x_b) > 1$. The associated line segment does not belong to $\mathcal{X}$, i.e. $\{tx_k + (1-t)x_b \mid t \in (0, \alpha)\} \not\subseteq \mathcal{X}$ and therefore $x_k \notin \ker\mathcal{X}$, contradicting our assumption.
$\Leftarrow$: The proof of the reverse direction is nearly identical. □

Remark. From Lemma 4 we see that the kernel of $\mathcal{X}$ is defined by cutting-planes tangent to the active constraint $g_i(x_b) = 1, x_b \in \partial\mathcal{X}$ as shown in Figure 1.

Although $\ker\mathcal{X}$ is a convex, semialgebraic set it is not straightforward to represent it within a semidefinite program. Determining if a convex, semialgebraic set is semidefinite-representable is an area of active research and there is not a systematic procedure for constructing the representation if one exists [13]. Instead, we provide sampling-based algorithms for finding outer and inner approximations of this set. If the outer approximation is empty, this is sufficient to conclude that the set $\mathcal{X}$ is not star-convex. Conversely, if the inner approximation is not empty this is sufficient to establish that $\mathcal{X}$ is star-convex. In the case that the outer approximation is not empty and the inner approximation is empty we cannot conclude anything about the star-convexity of the set.

A. Outer Approximation

We assume the existence of an oracle Sample($\partial\mathcal{X}$) which allows us to randomly sample points $x_b \in \partial\mathcal{X}$ and identify the set of active constraints $\mathcal{I} = \{i \mid i \subseteq [m], g_i(x_b) = 1\}$. From Lemma 4, each sample defines a cutting plane satisfied by $\ker\mathcal{X}$. We collect these constraints to form an outer approximation $\mathcal{K}_o \supseteq \ker\mathcal{X}$. If at any point, $\mathcal{K}_o = \emptyset$ (which can be determined using Farkas' Lemma) we terminate as this implies $\ker\mathcal{X} = \emptyset$. Algorithm 2 summarizes the method.

B. Inner Approximation

Consider finding a point $x_k \in \ker\mathcal{X}$ that maximizes a linear cost $c^T x_k$ where $c \in S^{n-1}$ (i.e. the support function of $\ker\mathcal{X}$). From Lemma 4, the resulting convex optimization problem requires set containment constraints.

$$\begin{aligned} \min_{x_k} \quad & -c^T x_k \\ \text{s.t.} \quad & \\ & -\nabla g_i(x)^T (x_k - x) \geq 0 \, \forall \, x \in \partial\mathcal{X}_i, i \in [m] \end{aligned} \tag{18}$$

Algorithm 2 Outer Approximation of ker$\mathcal{X}$

Input: $\mathcal{X} \subset \mathbb{R}^n$, Number of samples n_s
Output: Outer Approximation $\mathcal{K}_o \supseteq$ ker$\mathcal{X}$
  $\mathcal{K}_o \leftarrow \mathbb{R}^n$
  for $j = 1$ to n_s **do**
    $x_b, \mathcal{I} \leftarrow$ Sample($\partial \mathcal{X}$)
    $\mathcal{K}_o \leftarrow \mathcal{K}_o \bigcap \{x \,|\, \nabla g_i^T(x_b)(x - x_b) \leq 0, i \in \mathcal{I}\}$
    if ($\mathcal{K}_o = \emptyset$) **then**
      return $\mathcal{K}_o$
  return $\mathcal{K}_o$

We replace the set containment conditions with sufficient conditions using the generalized $\mathcal{S}$-procedure.

Optimization Problem: FindSupport$(\mathcal{X}, c)$

$$\begin{aligned} \min_{x_k, \lambda_j^{(i)}(x)} \quad & -c^T x_k \\ \text{s.t.} \quad & \\ -\nabla g_i(x)^T(x_k - x) - \sum_{j=1}^{m} \lambda_j^{(i)}(x)(1 - g_j(x)) &\in \textstyle\sum[x], i \in [m] \\ \lambda_j^{(i)}(x) \in \textstyle\sum[x],\ i \in [m], j &\in [m] \setminus i \end{aligned} \tag{19}$$

For a given direction $c \in S^{n-1}$ this program lower bounds the support function of ker$\mathcal{X}$. If the problem is feasible, the minimizing argument x_k belongs to ker$\mathcal{X}$ (though it may lie in the interior) and therefore $\mathcal{X}$ is star-convex. If the problem is infeasible we cannot make any conclusions about the star-convexity of $\mathcal{X}$. By solving for random directions $c_i \in S^{n-1}, i \in [n_s]$ the convex hull of the points x_k provides an inner approximation of the kernel as given by Algorithm 3.

Algorithm 3 Inner Approximation of ker$\mathcal{X}$

Input: $\mathcal{X} \subset \mathbb{R}^n$, Directions $c_i \in S^{n-1}, i \in [n_s]$
Output: Inner Approximation $\mathcal{K}_i \subseteq$ ker$\mathcal{X}$
  $\mathcal{K}_i \leftarrow \emptyset$
  for $i = 1$ to n_s **do**
    $x_k \leftarrow$ FindSupport($\mathcal{X}, c_i$)
    if FindSupport($\mathcal{X}, c_i$) = Infeasible **then**
      return $\mathcal{K}_i = \emptyset$
    $\mathcal{K}_i \leftarrow$ conv($\mathcal{K}_i, x_k$)
  return $\mathcal{K}_i$

C. Kernel of Unions and Intersections

Given sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$ and their kernels, we can find inner approximations of the kernel of their intersection and union using the following lemma.

Lemma 5. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$. Then the following holds:* [2]

$$\ker(\mathcal{A} \cap \mathcal{B}) \supseteq \ker\mathcal{A} \cap \ker\mathcal{B} \tag{20}$$

$$\ker(\mathcal{A} \cup \mathcal{B}) \supseteq \ker\mathcal{A} \cap \ker\mathcal{B} \tag{21}$$

[2]Simple examples can be constructed to show that there is no relation between ker$(\mathcal{A} \cap \mathcal{B})$ and ker$(\mathcal{A} \cup \mathcal{B})$ in general.

TABLE I
PERCENT ERROR OF OUTER APPROXIMATIONS OF EXAMPLES A-C

Example	Degree	s	$-\det P$	trP^{-1}	l_1
A	4	11.9	35.1	40.0	18.3
A	6	1.4	8.3	10.0	12.8
B	4	17.7	31.1	35.0	37.3
B	6	4.9	9.7	14.0	17.7
C	4	2.6	20.1	21.2	15.3
C	6	0.6	7.2	7.4	11.0

Proof. See appendix.

Thus if $\mathcal{A}, \mathcal{B}$ are star-convex and have kernels that intersect, their union and intersection is also star-convex. This is useful for establishing star-convexity without resorting to numerical algorithms.

VI. EXAMPLES

We evaluate Algorithm 1 on various examples and compare the results to the existing heuristics reviewed in Section III.[3] We focus our comparison on outer approximations as more heuristics apply to this case. We use percent error as our metric, calculated as $100 \times \frac{\text{vol}\mathcal{F}_o - \text{vol}\mathcal{X}}{\text{vol}\mathcal{X}}$ where $\mathcal{F}_o$ is the outer approximation of $\mathcal{X}$. We first consider three examples from the literature. For both 4th-order and 6th-order polynomials, our algorithm yielded the tightest outer approximation as shown in Table I.[4] Next we consider 100 randomly generated convex polytopes in $\mathbb{R}^2$. In the majority of cases, our heuristic yielded the tightest outer approximation as shown in Table II.

A. Polynomial matrix inequality [3]

$$\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, \begin{bmatrix} 1 - 16x_1x_2 & x_1 \\ x_1 & 1 - x_1^2 - x_2^2 \end{bmatrix} \succeq 0\}$$

Using Algorithms 2 and 3 we find the exact kernel ($\mathcal{K}_o = \mathcal{K}_i = \text{conv}\{\pm(-0.1752, 0.3335), \pm(0.1268, 0.2213)\}$) as shown in Figure 1. Figure 2 shows the 4th-order inner and outer approximation obtained with Algorithm 1.

B. Discrete-time stabilizability region [3],[1]

$$\begin{aligned} \mathcal{X} = \{x \in \mathbb{R}^2 \,|\, 1 + 2x_2 \geq 0, 2 - 4x_1 - 3x_2 \geq 0, \\ 10 - 28x_1 - 5x_2 - 24x_1x_2 - 18x_2^2 \geq 0, \\ 1 - x_2 - 8x_1^2 - 2x_1x_2 - x_2^2 - 8x_1^2x_2 - 6x_1x_2^2 \geq 0\} \end{aligned}$$

The set contains the origin in its kernel. Figure 2 shows the 4th-order inner and outer approximation obtained with Algorithm 1. Figure 3 shows the 6th-order approximations obtained with each objective. For the l_1 approximation we also show the bounding box $\mathcal{B}$ as given in [1].

[3]For the bounding box $\mathcal{B}$ required by the l_1 objective, we used the smallest hyperrectangle $\mathcal{B} \supseteq \mathcal{X}$ unless noted otherwise.

[4]We forego comparing 2nd-order polynomials as the determinant maximization objective exactly minimizes volume in this case.

TABLE II

INSTANCES IN WHICH OBJECTIVE OBTAINED SMALLEST ERROR

Deg.	# Trials	s	$-\det P$	$\text{tr}P^{-1}$	l_1
4	100	73	13	0	14
6	100	98	0	0	2

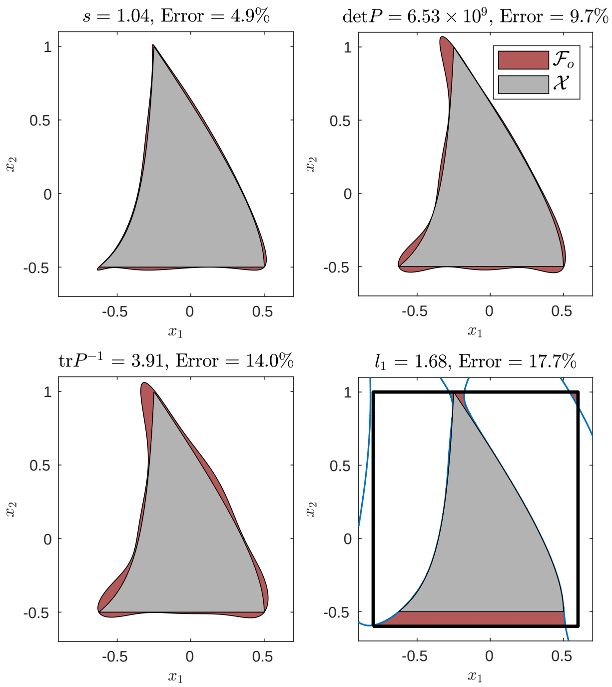

Fig. 3. 6th-order outer approximations of example B

C. *General Non-Convex Set* [5]

$$\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, x_2 \leq 0.5x_1^2\}$$

This set is star-convex but does not contain the origin in its kernel. In applying Algorithm 1 we translated the set to the Chebyshev center of its kernel $x^* = (1.39, 0.35)$.

D. *Convex Polytopes*

We generate 100 random convex polytopes in $\mathbb{R}^2$ with their Chebyshev center at the origin. We find outer approximations using the different objectives. Table II lists the number of times each objective obtained the smallest percent error relative to the other objectives for a given polytope.

E. *Implementation Details*

YALMIP [14] and MOSEK [15] were used to solve the SOS programs.[5] In solving (14), we used polynomials $\lambda(x), \mu(x)$ with degree equal to that of the polynomial $f(x)$.

VII. CONCLUSIONS

An algorithm for finding approximations of star-convex semialgebraic sets using sum-of-squares optimization was proposed. The algorithm relies on a novel objective which minimizes the scaling necessary to transform an inner approximation into an outer approximation of the set. Numerical examples demonstrated this objective often finds tighter approximations compared to existing heuristics.

[5]Supporting code will be released upon publication.

APPENDIX

A. *Proof of Lemma 3*

Recall the Hausdorff distance between two compact, convex sets can be written in terms of their support functions.

$$\begin{aligned} d_H(s\mathcal{F}, \mathcal{F}) &= \max_{c \in S^{n-1}} |\sigma_{s\mathcal{F}}(c) - \sigma_{\mathcal{F}}(c)| && (22)\\ &= \max_{c \in S^{n-1}} |s\sigma_{\mathcal{F}}(c) - \sigma_{\mathcal{F}}(c)| && (23)\\ &= (s-1) \cdot \max_{c \in S^{n-1}} \sigma_{\mathcal{F}}(c) && (24)\\ &= (s-1) \cdot \max_{x \in \mathcal{F}} \|x\|_2 && (25) \end{aligned}$$

B. *Proof of Lemma 5*

1) $\ker(\mathcal{A} \cap \mathcal{B}) \supseteq \ker\mathcal{A} \cap \ker\mathcal{B}$: Let $l(x, y) = \{\lambda x + (1-\lambda)y \,|\, \lambda \in [0, 1]\}$ for some $x \in \ker\mathcal{A} \cap \ker\mathcal{B}$ and $y \in \mathcal{A} \cap \mathcal{B}$. As $x \in \ker\mathcal{A}, y \in \mathcal{A} \implies l(x, y) \subseteq \mathcal{A}$ and similarly, $x \in \ker\mathcal{B}, y \in \mathcal{B} \implies l(x, y) \subseteq \mathcal{B}$, we see that $x \in \ker(\mathcal{A} \cap \mathcal{B})$.□

2) $\ker(\mathcal{A} \cup \mathcal{B}) \supseteq \ker\mathcal{A} \cap \ker\mathcal{B}$: Let $l(x, y) = \{\lambda x + (1-\lambda)y \,|\, \lambda \in [0, 1]\}$ for some $x \in \ker\mathcal{A} \cap \ker\mathcal{B}$ and $y \in \mathcal{A} \cup \mathcal{B}$. For the case when $y \in \mathcal{A}$, then $x \in \ker A \implies l(x, y) \subseteq \mathcal{A} \implies l(x, y) \subseteq \mathcal{A} \cup \mathcal{B}$. Similarly, for the case when $y \in \mathcal{B}$, then $x \in \ker B \implies l(x, y) \in \mathcal{B} \implies l(x, y) \subseteq \mathcal{A} \cup \mathcal{B}$. Therefore $x \in \ker(\mathcal{A} \cup \mathcal{B})$.□

REFERENCES

[1] F. Dabbene, D. Henrion, and C. M. Lagoa, "Simple approximations of semialgebraic sets and their applications to control," *Automatica*, vol. 78, pp. 110–118, 2017.

[2] A. Magnani, S. Lall, and S. Boyd, "Tractable fitting with convex polynomials via sum-of-squares," in *Proceedings of the 44th IEEE Conference on Decision and Control*, pp. 1672–1677, 2005.

[3] D. Henrion and J.-B. Lasserre, "Inner approximations for polynomial matrix inequalities and robust stability regions," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1456–1467, 2012.

[4] A. A. Ahmadi, G. Hall, A. Makadia, and V. Sindhwani, "Geometry of 3d environments and sum of squares polynomials," in *Robotics: Science and Systems*, 2017.

[5] V. Cerone, D. Piga, and D. Regruto, "Polytopic outer approximations of semialgebraic sets," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pp. 7793–7798, 2012.

[6] J. Guthrie and E. Mallada, "Outer approximations of minkowski operations on complex sets via sum-of-squares optimization," in *2021 American Control Conference (ACC)*, pp. 2367–2373, 2021.

[7] J. Guthrie, M. Kobilarov, and E. Mallada, "Closed-form minkowski sum approximations for efficient optimization-based collision avoidance," in *2022 American Control Conference (ACC)*, 2022.

[8] M. Jones and M. M. Peet, "Using sos for optimal semialgebraic representation of sets: Finding minimal representations of limit cycles, chaotic attractors and unions," in *2019 American Control Conference (ACC)*, pp. 2084–2091, 2019.

[9] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, March 2004.

[10] J.-B. Lasserre, "A generalization of löwner-john's ellipsoid theorem," *Mathematical Programming*, vol. 152, 08 2014.

[11] T.-C. Wang, S. Lall, and M. West, "Polynomial level-set methods for nonlinear dynamical systems analysis," in *Proc. Allerton Conf. on Communication, Control, and Computing, Allerton, IL*, Citeseer, 2005.

[12] P. Parrilo, *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*. PhD thesis, California Institute of Technology, 2000.

[13] J. W. Helton and J. Nie, "Semidefinite representation of convex sets," *Mathematical Programming*, vol. 122, no. 1, pp. 21–64, 2010.

[14] J. Lofberg, "Yalmip : a toolbox for modeling and optimization in matlab," in *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No.04CH37508)*, pp. 284–289, Sep. 2004.

[15] M. ApS, *The MOSEK optimization toolbox for MATLAB manual. Version 8.1.*, 2017.